Stochastic master-equation approach to aggregation in freeway traffic

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In analogy to the usual aggregation phenomena, such as the formation of liquid droplets in a supersaturated vapor, the nucleation, growth, and condensation of car clusters are considered in a one-lane freeway traffic flow model. The clustering behavior (known as congestion) in an initially homogeneous traffic stream is described by a master equation. The construction of the stochastic equation is given as well as its relationship to other dynamical models. Numerical experiments in heavy traffic with well-explained transition probabilities show the transition from the initial free particle situation (free jet of vehicles) to the final congested cluster state, where one big aggregate of cars is formed. The results are presented analytically in dependence of the stable cluster size from car concentration and numerically as stochastic trajectories. [S1063-651X(97)05109-X]

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I. INTRODUCTION

Fluid-dynamical approaches to traffic flow were developed long ago starting in the 1950s by Lighthill and Whitham. Since the pioneering work of Prigogine and Herman [1] on the kinetic theory of vehicular traffic, cars have been considered as interacting particles. Depending on the number of cars on the road (the density of cars) bound states (so-called car clusters) may become possible. If the density is small the free flow of nearly noninteracting particles is dominant. The flux rate increases linearly with the density. If the density of cars exceeds some critical value a jamming transition takes place. This phase transition separates the low-density situation in which all cars move independently at maximal speed from the high-density region in which the formation of car aggregates as bound states reduces the average velocity of cars. The flux rate is decreasing with increasing car density. In a one-dimensional situation (single-lane traffic) all cars are stopped if the road is crowded with cars (density equals one). The curves in the flux-density space are known as a fundamental diagram.

Based on several approaches [1–6] such as a cellular automaton model for freeway traffic by Nagel and Schreckenberg [7] and others [8–11] the fundamental diagram shows clearly the phase transition from the free-jet situation (no aggregation effect) to the car cluster regime with start-stop waves. It has been shown by Kerner and Konhäuser [12,13] that for high densities a car cluster can spontaneously appear in which the average velocity of cars is considerably lower than in the initial flow and outside the cluster. Surprisingly already simple particle hopping models on the basis of cellular automata rules can lead to realistic space-time car dynamics and fit into the general context of traffic flow theory as pointed out by Nagel [14]. This seems to be the case for both elementary types of lattice rules (single-lane traffic, one kind of vehicle) [7] and more sophisticated variants (two-lane traffic with lane changing, different kinds of vehicles) [10]. A description in terms of lattice rules also permits a discussion of critical phenomena of traffic-jam formation. The system self-organizes in such a way that the outflow from a large car cluster is a critical state of maximum throughput. Numerical results and phenomelogical theory based on random walks show that slow perturbations in the outflow of a big car cluster lead downstream to traffic jams of all sizes. The creation of emergent car clusters in a driven, randomly perturbed system is a particulary good example for criticality [15–17].

The aggregation of particles (e.g., the emergence of car clusters in traffic flow) out of an initially homogeneous situation is well known in physics. Depending on the system under consideration and its control parameters the cluster formation in a supersaturated (unstable) situation has been observed in nuclear physics as well as in other branches. We mention the well-known example of condensation (formation of liquid droplets) in an undercooled water vapor. The formation of bound states as an aggregation process is related to self-organized phenomena [18,19]. Self-organization by nonlinear irreversible processes is well known not only in physics but in other branches such as biology and sociology; see, e.g., [18,20].

Dynamical models for cluster formation based on stochastic methods (master equation, Langevin equation, Fokker-Planck equation) have, to our knowledge, not been exploited so far in traffic theory. We mention Ref. [17] where random-walk arguments have been used to discuss lifetime distributions of jams. It is the aim of the present paper to give a stochastic description of jam formation using the master-equation approach. The main point is to construct the transition probabilities for the jump processes.

II. DYNAMICS OF A SPONTANEOUS TRAFFIC JAM

The possible states of a highway traffic at varying densities are well known; let us briefly discuss them in review. In contrast to the phase diagram of a van der Waals gas we have

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a *gas phase* at low densities: the particles are considered to be noninteracting (ideal gas flow); an initially homogeneous flow will remain stable; the flow-density curve in the fundamental diagram is linear, whose slope is the mean velocity. The relationship between mean velocity and density can be described either by a *security distance* law, or by an *optimal velocity* law.

On the other hand, at high densities we find the *liquid phase*: the particles interact strongly (condensed flow) and an initially homogeneous flow will also remain stable. The mean velocity and the flow reduce as the density gets greater, and reach zero at a finite maximal value for density (i.e., at a minimal value for the intervehicle distance).

At intermediate densities there is a region of phase separation between both stable phases. Since the particles interact episodically an initially homogeneous flow becomes unstable. The birth of one or several aggregates as clustering by collisions takes place. The car clusters can either disappear by concurrence or rapidly reach very high local densities and very low speeds for the trapped vehicles. The clusters as binding states move backwards with a speed that is directly dependent on the rate of incoming cars, i.e., the upstream flow. The general behavior is an alternation of free and congested areas, often called *stop-and-go waves*, or *stop-start waves*. There are two critical densities in the vicinity of which metastabilities and bifurcations may occur.

Keeping in mind our goal of building a master equation that could describe the stochastic evolution of the aggregation phenomenon, we have now to find a simplified but still realistic definition for a traffic jam. Since many results, for example [12,13], tell us that, in a new-born jam, the convergence towards very high density and corresponding very low individual speed is significantly higher than that of the *size* of the jam towards its stationary size, we will give the following definition of a car cluster: A cluster of size n is an aggregate of n vehicles whose individual speed is zero, and whose front-to-rear (bumper-to-bumper) distance is zero. That we have set the minimal allowed distance between two vehicles to zero does not reduce generality, since we can consider the *effective length* of a car as a minimal distance added to the real length. However, the zero-speed hypothesis is a rough simplification; a useful generalization of this definition would then be to allow density as an additional variable (besides the size) for the characterization of the cluster. Our definition of a single jam forming a queue of cars having zero velocity is in agreement with the work of Nagel and Paczuski [17]. A further simplification performed in this model is that we will allow only one cluster at a time. Other works [21] have shown that the total number of cars blocked on the road is practically independent of the number of coexisting clusters, so that our results should not be too strongly affected by this restriction.

III. FOLLOW-THE-LEADER BEHAVIOR

In order to allow a description of the homogeneous flow behavior, we have to choose a relationship between the speed of a given vehicle and the distance to its leader. As this particular problem is not the essential point of this paper, and should be further discussed and justified in confrontation with experimental data, our choice was that of a sufficiently simple, but still realistic model, namely, a *sigmoidal optimal velocity* curve. Sigmoidal functions have a characteristic shape that starts at one value and rises smoothly to another value with a single inflection point. There are many algebraic expressions to represent such curves, but in applications such as biological growth laws, only a few of them are encountered frequently [20]. We mention the hyperbolic tangent, which has been used by Bando *et al.* [21–23] as the optimal velocity curve. We make another choice, sometimes known as the Hill function of second order, analytically written as

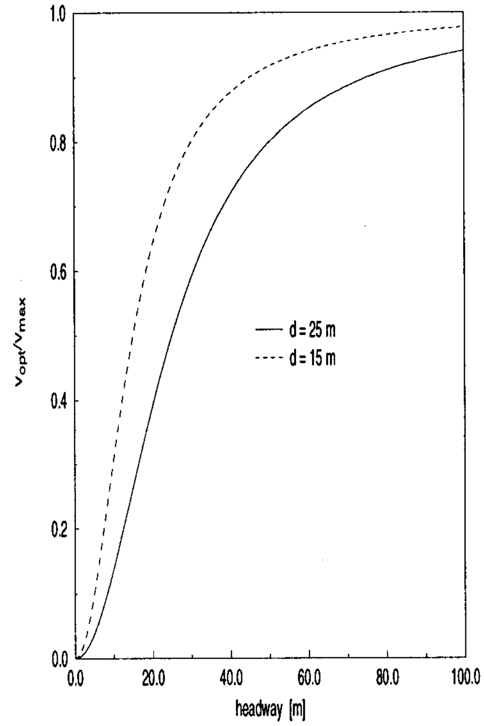

FIG. 1. Sigmoidal curves for optimal velocity showing the desired speed (dimensionless quantity $v_{\mathrm{opt}}/v_{\mathrm{max}}$) in dependence of the headway h to the next car.

$$v_{\mathrm{opt}}(h)=v_{\mathrm{max}}\frac{h^2}{d^2+h^2}, \tag{1}$$

where h is the headway (bumper-to-bumper with effective length), v_{max} is the maximal speed allowed, and d is a positive control parameter, which can be seen as a characteristic headway for the transition between noninteracting and interacting phases.

This particular form [Eq. (1)] of the optimal velocity curve (Fig. 1) has the advantage of allowing easy analytical treatment for the following developments, and of being consistent with the limit $v_{\mathrm{opt}}(0)=0$; the fact that the upper limit $h\to\infty$ is not consistent with the periodic boundaries does not appear to be problematic.

Besides the movement with an optimal velocity behavior the car-car interaction has to be considered. In comparison to a granular particle flow in narrow pipes [24] the clustering of cars is driven by inelastic collisions. When two particles collide inelastically (two cars reach each other without accident), their velocities change so that the faster car adopts the velocity of the slower one, and hence they remain close to one another. The proposed mechanism for the formation of aggregates allows us to derive a stochastic equation describing the creation and evolution of a car cluster of size n.

IV. THE MASTER EQUATION

The master equation is a differential equation describing the time evolution of the probability distribution of a set of random variables for a stochastic dynamical system [25,26] that follows a Markov process, i.e., for which the transition probabilities depend on the actual state but not on the past states. It is expressed in terms of transition rates between the reachable states of the system. Mostly the description of dynamical stochastic processes is limited to linear models such as one-dimensional birth-death equations or random walks on a line with unknown parameters [25,26,17]. Our approach to aggregation in freeway traffic requires not only the knowledge of all jump processes and their generally nonlinear transition probabilities but also the knowledge of all parameters used in the model. Since such quantities can rarely be determined from fundamental equations the parameters must be observable from experimental data.

In our particular case, the only random variable is the size n of the cluster. We will furthermore accept that only one car at a time can go into or come out of the jam; this means that we do not consider the merging or splitting of aggregates, which is a consequence of our choice of a one-cluster system. As a consequence, only two kinds of transitions are allowed: a growth transition $n \rightarrow n+1$ and a decay transition $n \rightarrow n-1$, to which we associate a growth transition rate w_+, and a decay transition rate w_-, respectively. These rates depend in general on the variable n; they could also be time dependent if we would consider the possibility of an external control on them (e.g., through the action of traffic lights or a temporary adaptation of legal speed), a case that will not be treated here but that should be a necessary step towards practical applications, such as technical or legal measures that can be taken to prevent jamming. We can now write our master equation,

$$\frac{\partial}{\partial t} p(n,t) = w_+(n-1)p(n-1,t) + w_-(n+1)p(n+1,t) - [w_+(n)+w_-(n)]p(n,t), \tag{2}$$

which can be used to determine numerically the temporal evolution of the stochastic variable $n(t)$, the probability $p(n,t)$ to find a size-n jam at time t, the mean value as well as the variance for the random variable n, and hence the desired evolution of the flow. Nevertheless, in order to deal with this equation, we still need the transition rates, which have to emerge from phenomenological considerations.

V. RELAXATION TIMES AND TRANSITION RATES

A very important parameter in many highway traffic models is the relaxation time, which describes the driver's technical-psychological fastness of adaptation to the actual downstream state of the flow. Associated with an optimal velocity model, it is the time constant for the exponentially asymptotic adaptation from his actual speed to the optimal speed required for his actual headway. However, in these models it is generally assumed, implicitly, that time constants for speeding up and for slowing down are identical, an assumption that has obviously no reasonable grounds, either technically or psychologically. We shall on the contrary assume that *the braking time constant is much greater than the accelerating one*. Indeed, as a driver approaches the rear of the jam, he does not realize immediately that he will have to come to a real stop, therefore he will use his brakes very late and very strongly. In the opposite case, namely, when the driver sees again the free flow in front of him, he will have adapted his speed, which is zero at the time, to the mean speed of the free flow, after a time that can be approximated by a *constant*.

As a consequence of these statements, we are able to define the transition rates in a very simple way. Let us discuss both cases separately. Considering the free flow upstream of the jam, we remember that we assumed it was homogeneous, with an average headway h and speed $v_{\mathrm{opt}}(h)$. At time t, the car directly behind the jam has to drive a distance h, and, if we take the extreme assumption that the decelerating time constant is zero, it needs an average time $h/v_{\mathrm{opt}}(h)$ to get into the jam. Hence, the probability, at time $t+dt$, that the car has collided (''jumped'') into the jam is $dt\, v_{\mathrm{opt}}(h)/h$. The transition rate, which is defined per time unit, will therefore be

$$w_+(h) = \frac{v_{\mathrm{opt}}(h)}{h}. \tag{3}$$

On the other hand, as the head car of the jam can come free again, it will need the average time τ, and hence the corresponding transition rate is simply

$$w_-(h) = \frac{1}{\tau}. \tag{4}$$

Now, based on the conservation law of finite systems that all cars are moving on a street with periodic boundary conditions, with the help of the following easy relation between the headway h and the size of the cluster n,

$$h(n) = \frac{L - N l_0}{N - n}, \tag{5}$$

where L is the total length of the road, N the total number of cars, and l_0 the effective length of a single car, we finally find (see Fig. 2) the different forms of the curve $w_+(n)$ for different densities $\varrho = N/L$. For $w_-(n)$ we still have a straight line $w_-(n) = w_- = 1/\tau = \mathrm{const}$. The general parameters for the figures are chosen to be realistic enough, but they are not empirically justified; however, we will see that these realistic parameters will lead to a realistic flow-density behavior.

The curves shown in Fig. 2 allow us to describe the different behaviors connected with different relative values of the control parameters. The first case to be examined should be the case where $1/\tau$ is bigger than the maximum value for w_+, with the condition $\tau < 2d/v_{\mathrm{max}}$, since this case leads to a stable homogeneous flow for the whole range of densities; however, for realistic values of parameters, for example, $d = 20$ m, $v_{\mathrm{max}} = 40$ m/s, this situation will not occur, as the

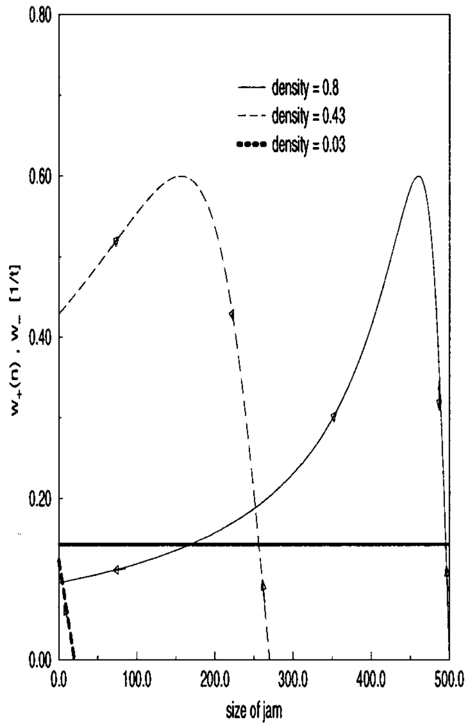

FIG. 2. Analytical curves for the transition rates (decay rate: horizontal line; growth rates: curves for undercritical, overcritical and very high densities) for three different regimes. The parameters are $d=20$ m, $v_{\max}=40$ m/s, $l_0=8$ m, $\tau=7$ s, $L=5000$ m.

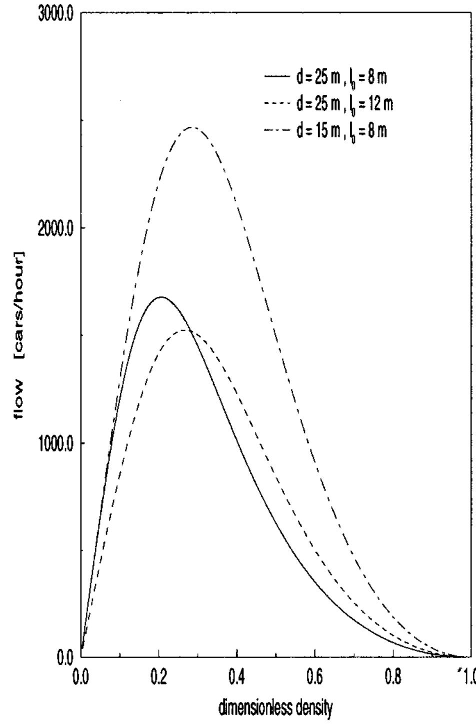

FIG. 3. Examples of flow-density curves obtained. The computed fundamental diagram shows the validity of our approach and agrees with measured data curves.

time constant should be less than 1 s. For the opposite case, the density range appears to be divided into three parts by two critical values. Below the lower critical value, the growing transition will always be less probable than the decay transition, and hence any initial perturbation will disappear. Here we find again that for low densities, the homogeneous flow is stable. For intermediate densities, we find one equilibrium size, corresponding to the intersection point $w_+(n)=w_-$, which is a *stable* stationary size of the cluster. This means that, after a long time, the system is a stationary two-phase system, as described by several authors, with the cluster moving backwards through a homogeneous flow. Now, for high densities, there are *two* equilibrium sizes, the greater one being stable and the smaller one being unstable; thus, for initially homogeneous or slightly perturbed flows, in other words, if the initial size of the perturbation is less than the unstable critical size, the perturbation will disappear; we then find the homogeneous congested flow described in general highway measurements. On the other hand, if the initial cluster is bigger than this critical size, its size will increase even more, until it reaches the stable stationary value. This behavior at high densities can be seen as a metastability of the homogeneous flow, since it needs a minimal perturbation to get unstable.

From the knowledge of equlibrium cluster sizes in dependence of the density, we can already calculate analytically the mean value of the general flow at all densities. In other words we can find the *fundamental diagram*, as plotted in Fig. 3, as a first test of the validity of our model. We see that the shape is like most analytical and empirical curves. We see also the influence of the parameter d, which can be seen as the ''range'' of the interaction, and of the parameter l_0, which is the size of the ''particle.''

VI. JAM DYNAMICS: ANALYTICAL AND NUMERICAL RESULTS

The master equation (2), with its associated transition rates given by Eqs. (3) and (4), with the help of Eq. (5), allows us to calculate analytically and numerically the time evolution for the mean value of the size, as well as of its variance, for different sets of parameters, and in particular for different values of the density. In order to find an analytical expression for the time evolution of the mean value $\langle n\rangle(t)=\Sigma_{n=0}^{N}np(n,t)$ from the master equation (2), we have to perform the approximation $\langle w_+(n)\rangle \simeq w_+(\langle n\rangle)$, which leads to the following form:

$$\frac{d}{dt}\langle n\rangle \simeq w_+(\langle n\rangle)-\frac{1}{\tau}, \qquad (6)$$

with stationary solutions of the mean car cluster size

$$\langle n\rangle_{\text{stat}}=N-\frac{L/l_0-N}{v_{\max}\tau/2l_0\pm\sqrt{(v_{\max}\tau/2l_0)^2-(d/l_0)^2}} \qquad (7)$$

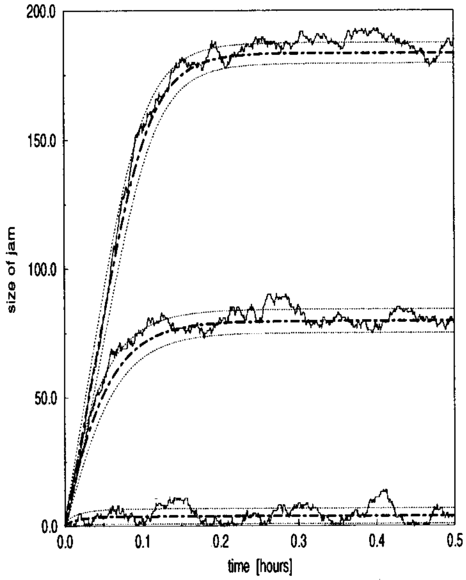

FIG. 4. Examples of stochastic trajectories as solutions of the master equation for intermediate densities (from bottom to top the total number of cars increases $N=24,100,200$) compared with numerical solutions for the mean cluster size and its variance.

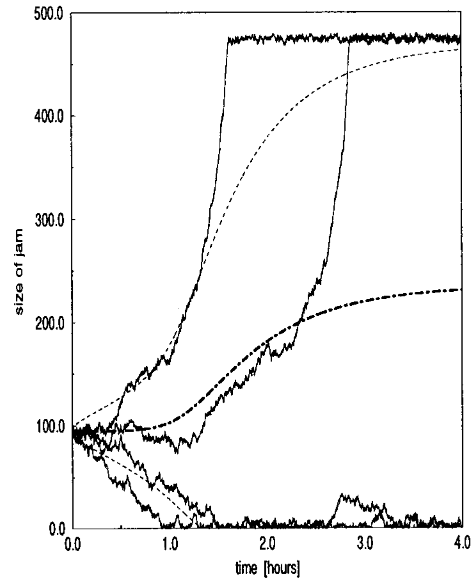

FIG. 5. Stochastic trajectories starting with critical initial size (unstable cluster size) at a very high density 0.68 (total number of cars $N=480$) together with the mean value.

in the range of heavy traffic with overcritical densities. Equation (6) correctly describes the dynamics of the system at far-from-critical densities, and have basically the right sta tionary solutions. In parallel to this, we simulate particular stochastic trajectories, which are in accordance with the results computed from the derived master equation (see Figs. 4 and 5).

For far-from-critical densities, the evolution is quasideterministic and the fluctuations are small. On the other hand, the high density trajectories show the bifurcation at a high density around the critical initial jam size. The initially inhomogeneous flow can have a long lifetime, and then bifurcate either into a homogeneous congested flow ($n \rightarrow 0$), or into a stable state where many cars are blocked into a jam ($n \rightarrow n_{\text{stable}}$). For realistic values of the parameters, we see that this big cluster contains almost all the cars. As a result, if we take into account the fluctuations of the system, we can see that both final states are equivalent. Indeed, in the homogeneous flow, the cars move at a very low speed, and could be considered—with a less restrictive definition than ours for the cluster (cf. Sec. II), allowing nonzero particle velocity—being inside a jam that covers the road completely, with fluctuations around homogeneity. These fluctuations could be analyzed, for example, as the propagation of ''holes'' (free spaces) in a homogeneous medium (see also [27]). However, these holes do not describe a discrete system anymore—having no finite fixed size—and therefore do not allow for symmetry between very high and very low densities.

VII. CONCLUSIONS

In conclusion we emphasize that the model described in this paper is an approach to traffic problems based on a one-dimensional master equation. By focusing on the jamming phenomenon and on its essentially stochastic properties, we have been able to describe the general behavior of the system very realistically under very simple hypotheses and with very small computing power. In particular, our fundamental diagram (Fig. 3) shows its habitual shape. Of course, many improvements and generalizations are possible and necessary. In particular, a more subtle definition of the car cluster (cf. Sec. II) would be helpful in describing the behavior of the system at very high densities (cf. discussion at end of Sec. VI), but also at low densities—since in that case we do not expect the cars to come to a real stop by agglomerating. The decay transition probability [Eq. (4)] could be defined in a more precise way, and in particular should be density dependent. The parametrization should be investigated and quantitative results compared with real life. Allowing coexistence of several clusters and overtaking (which could be done using an analogy with tunnel effect), examining the influence of traffic lights (which create traffic jams artificially) and of crossroads (which create jams by external stochastical process) should be the first steps to generalization and applications.

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